Shape Avoiding Permutations

Ron M. Adin*‡ Yuval Roichman†‡

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Abstract

Permutations avoiding all patterns of a given shape (in the sense of Robinson-Schensted-Knuth) are considered. We show that the shapes of all such permutations are contained in a suitable thick hook, and deduce an exponential growth rate for their number.

1 Introduction

1.1 Outline

The Robinson-Schensted(-Knuth) correspondence is a bijection between permutations in S_n and pairs of standard Young tableaux of the same shape (and size n). This common shape is called the *shape* of the permutation. A permutation $\pi = (\pi_1, \ldots, \pi_n)$ in S_n *avoids a permutation* $\sigma = (\sigma_1, \ldots, \sigma_m)$ in S_m if there is no subsequence $(\pi_{i_1}, \ldots, \pi_{i_m})$ of π such that $\pi_{i_j} > \pi_{i_k}$ iff $\sigma_j > \sigma_k$ $(\forall j, k)$. π *avoids a shape* μ if it avoids all the permutations of shape μ.

This paper deals with the relation between the property "π does not avoid a given shape μ" and the property "$\lambda = shape(\pi)$ contains μ as a subshape". It turns out that, in general, neither of these properties implies or contradicts the other; but in certain important cases, such implications do hold. These cases include, e.g., rectangular shapes and hook shapes (either

*Department of Mathematics and Computer Science, Bar-Ilan University, Ramat-Gan 52900, Israel. Email: radin@math.biu.ac.il

†Department of Mathematics and Computer Science, Bar-Ilan University, Ramat-Gan 52900, Israel. Email: yuvalr@math.biu.ac.il

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for λ or for μ). These positive results are then applied to get asymptotic bounds related to the Stanley-Wilf conjecture on pattern-avoiding permutations (see Corollaries 4 and 5 in Subsection 1.2, and Subsection 7.2). Use is made of the Berele-Regev asymptotic evaluation of the number of standard Young tableaux contained in a “thick hook”.

The rest of the paper is organized as follows. The main results are listed in Subsection 1.2. Standard notations and necessary background are given in Section 2. In Section 3 we motivate our investigation by a “false conjecture”. In Section 4 we show that this “false conjecture” is correct for rectangular shapes. Using this knowledge we consider the general case in Section 5. Families of shapes, for which an exact evaluation may be obtained, are presented in Section 6. Section 7 concludes the paper with final remarks and open problems.

1.2 Main Results

For rectangular shapes the following holds.

Theorem 1. *If π is a permutation of rectangular shape (m^k), and μ is an arbitrary shape, then:*
μ is the shape of some subsequence of π if and only if $\mu \subseteq (m^k)$.

See Theorem 4.1 below.

Using Theorem 1 we prove the following general result.

Theorem 2. *For any permutation π in S_n and any partition $\mu = (\mu_1, \dots, \mu_k)$ of m :*
If $(\mu_1^k) \subseteq$ shape(π) then μ is the shape of some subsequence of π.

See Theorem 5.1 below.

For hook shapes a stronger result is proved.

Theorem 3. *Let m and k be positive integers and let $n \geq 4km$. Then for any hook $\mu = (m, 1^{k-1})$ and any permutation π in S_n :*
π has a subsequence of shape μ if and only if $\mu \subseteq$ shape(π).

See Theorem 6.1 below.

Denote by *avoid*$_n^\mu$ the size of the set of all μ-avoiding permutations in S_n. Combining Theorem 2 with the Berele-Regev asymptotic estimates [BR] the following bounds are proved.

Corollary 4. *For any fixed partition* $\mu = (\mu_1, \ldots, \mu_k)$,

$$\max\{ht(\mu), wd(\mu)\} \le \liminf_{n\to\infty} (avoid_n^{\mu})^{1/2n}$$

and

$$\limsup_{n\to\infty} (avoid_n^{\mu})^{1/2n} \le ht(\mu) + wd(\mu),$$

where the height of μ $ht(\mu) := k-1$, *and the width of* μ $wd(\mu) := \mu_1 - 1$.

See Corollary 5.2 below. It should be noted that this result is related to the Stanley-Wilf conjecture (see Subsection 7.2).

For hook shapes we have a sharper estimate.

Corollary 5. *For any pair of positive integers* m *and* k

$$\lim_{n\to\infty} (avoid_n^{(m,1^{k-1})})^{1/2n} = \max\{m-1, k-1\}.$$

See Corollary 6.5 below.

2 Preliminaries

Two classical partial orders on the set of partitions are considered in this paper. Let $\lambda = (\lambda_1, \ldots)$ and $\mu = (\mu_1, \ldots)$ be two partitions (not necessarily of the same number).

We say that μ is *contained* in λ, denoted $\mu \subseteq \lambda$, if

$$\mu_i \le \lambda_i \qquad (\forall i).$$

We say that μ is *dominated* by λ, denoted $\mu \preceq \lambda$, if

$$\sum_{j=1}^{i} \mu_j \le \sum_{j=1}^{i} \lambda_j \qquad (\forall i).$$

Clearly, $\mu \subseteq \lambda \Rightarrow \mu \preceq \lambda$.

The partition *conjugate* to λ is $\lambda' = (\lambda'_1, \ldots)$, where $\lambda'_i = \max\{j | \lambda_j \ge i\}$; i.e., the conjugate partition is obtained by interchanging rows and columns in λ.

Lemma 2.1. *[Md Ch. I (1.11)] If* λ *and* μ *are partitions of the same number* n *then*

$$\mu \preceq \lambda \Leftrightarrow \lambda' \preceq \mu'.$$

Corollary 2.2. *If λ and μ are partitions of the same number n, satisfying*

$$\mu \preceq \lambda \text{ and } \mu' \preceq \lambda'$$

then $\lambda = \mu$.

Define the *shape* of a sequence of integers to be the common shape of the two tableaux obtained via the Robinson-Schensted-Knuth correspondence. See [Sa §3.3, St §7.11]. The following theorem is well known.

Schensted's Theorem. *[Sc] For any partition λ and any permutation π of shape λ, the length of the longest increasing subsequence of π is equal to λ_1, and the length of the longest decreasing subsequence of π is equal to λ'_1.*

Schensted's Theorem was generalized by Greene.

Greene's Theorem. *[Gr] Let π be a permutation of shape $\lambda = (\lambda_1, \ldots, \lambda_t)$. Then, for all i:*

$$\sum_{j=1}^{i} \lambda_j = \text{ maximal size of a union of } i \text{ increasing subsequences in } \pi,$$

and

$$\sum_{j=1}^{i} \lambda'_j = \text{ maximal size of a union of } i \text{ decreasing subsequences in } \pi.$$

3 Motivation

Let μ be a partition of m, and let C^μ be the set of all permutations in S_m of shape μ. A permutation in S_n is a *μ-avoiding permutation* if it avoids all the permutations in C^μ; denote the set of these permutations by $Avoid_n^{\mu}$.

The only permutation in S_m having shape (m) is the identity permutation, i.e., a monotone increasing sequence. Schensted's Theorem, stated in the previous section, is thus equivalent to the following statement.

Fact 3.1. *For any pair of positive integers $m \le n$*

$$Avoid_n^{(m)} = \bigcup_{\{\lambda \vdash n \mid (m) \not\subseteq \lambda\}} C^\lambda,$$

and similarly for (1^m) instead of (m).

In other words, the set of permutations in S_n avoiding (m) is the union of all Knuth cells of shapes not containing (m). One may be tempted to think that this is a general phenomenon.

"False Conjecture" (First Version). *For any pair of positive integers $m \le n$ and any partition μ of m*

$$Avoid_n^{\mu} = \bigcup_{\{\lambda \vdash n \mid \mu \not\subseteq \lambda\}} C^{\lambda}.$$

Equivalently,

"False Conjecture" (Second Version). *For any permutation $\pi \in S_n$ of shape λ, the following two assertions hold:*

(1) For any partition $\mu \subseteq \lambda$ there exists a subsequence of π of shape μ.

(2) The shape of any subsequence of π is contained in λ.

Clearly, (1) is equivalent to the inclusion

$$Avoid_n^{\mu} \subseteq \bigcup_{\{\lambda \vdash n \mid \mu \not\subseteq \lambda\}} C^{\lambda},$$

while (2) is equivalent to the reverse inclusion

$$\bigcup_{\{\lambda \vdash n \mid \mu \not\subseteq \lambda\}} C^{\lambda} \subseteq Avoid_n^{\mu}.$$

Note that Greene's Theorem implies the weaker result that the shape of any subsequence of π is **dominated** by λ.

Unfortunately, the following examples show that both parts of the "False Conjecture" are false in general.

Example 3.2. *The permutation $\pi = (65127843)$ has shape $\lambda = (4, 2, 1^2)$, but has no subsequence of shape $\mu = (4, 1^3)$.*

Example 3.3. *The permutation $\pi = (25314)$ has shape $\lambda = (3, 1^2)$, but has a subsequence of shape $\mu = (2^2)$.*

Both examples can be extended to shapes λ of arbitrarily large size.

A central discovery in this paper is that the above "False Conjecture" is nevertheless correct in some important cases. This will be used to deduce asymptotic estimates.

4 Rectangular Shapes

A *rectangular shape* is a shape of the form (m^k), where m and k are positive integers. In this section we show that the "False Conjecture" is true whenever λ is a rectangular shape.

Theorem 4.1. *If π is a permutation of rectangular shape (m^k), and μ is an arbitrary shape, then:*
μ is the shape of some subsequence of π if and only if $\mu \subseteq (m^k)$.

In order to prove Theorem 4.1 we need the following consequence of Greene's Theorem.

Lemma 4.2. *Let π be a permutation of shape λ.*

(a) If π contains a disjoint union of k increasing subsequences of lengths $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_k$ then $(\ell_1, \ldots, \ell_k) \preceq \lambda$.

(b) If π contains a disjoint union of k decreasing subsequences of lengths $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_k$ then $(\ell_1, \ldots, \ell_k) \preceq \lambda'$.

Proof. By Greene's Theorem, for any $1 \leq i \leq k$

$$\sum_{j=1}^{i} \ell_j \leq \text{maximal size of a union of } i \text{ increasing subsequences of } \pi = \sum_{j=1}^{i} \lambda_j.$$

The proof of the second part is similar.

□

The following lemma characterizes permutations having rectangular shape.

Lemma 4.3.

(a) A permutation π has shape (m^k) if and only if the following two conditions are simultaneously satisfied:

(a1) π is a disjoint union of k increasing subsequences, each of length m.

(a2) π is a disjoint union of m decreasing subsequences, each of length k.

(b) If the above conditions hold, then each of the k increasing subsequences intersects each of the m decreasing subsequences in exactly one element.

Proof.

(a) Assume that π has shape λ and satisfies conditions (a1) and (a2) of the Lemma. By (a1) and Lemma 4.2(a), $(m^k) \preceq \lambda$. By (a2) and Lemma 4.2(b), $(k^m) \preceq \lambda'$. Also $|\lambda| = |(m^k)| = km$, so by Corollary 2.2, $\lambda = (m^k)$.

In the other direction: By Greene's Theorem, if π has shape (m^k) then it is the disjoint union of k increasing subsequences $\alpha_1, \ldots, \alpha_k$ of total size km. By Schensted's Theorem, each increasing subsequence of π has size at most m, and therefore $|\alpha_1| = \ldots = |\alpha_k| = m$. Similarly, π is a disjoint union of m decreasing subsequences $\beta_1, \ldots, \beta_m$ satisfying $|\beta_1| = \ldots = |\beta_m| = k$.

(b) Each increasing subsequence α_i intersects each decreasing subsequence β_j in at most one element, and since these km intersections cover all elements of π they are all nonempty.

□

Proof of Theorem 4.1. Let π be a sequence of shape $\lambda = (m^k)$. If μ is the shape of some subsequence of π then this subsequence contains an increasing subsequence of length μ_1. Therefore $\mu_1 \le \lambda_1 = m$. Similarly $\mu'_1 \le \lambda'_1 = k$, so that $\mu \subseteq (m^k)$.

In the other direction: By Lemma 4.3, π is a disjoint union of k increasing subsequences, of length m each, say $\alpha_1, \ldots, \alpha_k$ (enumerated arbitrarily). Similarly, π is a disjoint union of m decreasing subsequences, say $\beta_1, \ldots, \beta_m$ (of length k each). Also, each α_i intersects each β_j in a unique element; denote it by $P(i,j)$. Now let $\mu \subseteq (m^k)$, and define σ to be the subsequence of π consisting of all elements $P(i,j)$ with $j \le \mu_i$. We claim that σ has shape μ.

Indeed, σ intersects α_i in μ_i elements, and therefore (by Lemma 4.2(a)) $\mu \preceq \mathit{shape}(\sigma)$. Similarly, σ intersects β_j in μ'_j elements, and therefore (by Lemma 4.2(b)) $\mu' \preceq \mathit{shape}(\sigma)'$. Since $|\mathit{shape}(\sigma)| = |\mu|$ by definition, Corollary 2.2 implies that $\mathit{shape}(\sigma) = \mu$ and the proof is complete.

□

The following theorem is complementary.

Theorem 4.4. *If π is a sequence of shape λ and $(m^k) \subseteq \lambda$, then there exists a subsequence of π of shape (m^k).*

In other words: For any positive integers m and k

$$\mathit{Avoid}_n^{(m^k)} \subseteq \bigcup_{\{\lambda \vdash n \mid (m^k) \not\subseteq \lambda\}} C^\lambda.$$

Note that Example 3.3 shows that the converse of Theorem 4.4 is false.

Proof. Let π be a sequence of shape λ. By Greene's Theorem, π contains a disjoint union of k increasing subsequences of total size $\sum_{j=1}^{k}\lambda_j$. Denote this union by $\bar{\pi}$, and let $\mu := \mathit{shape}(\bar{\pi})$. Obviously, there are at most k parts in μ (i.e., $\mu = (\mu_1, \ldots, \mu_k)$ with $\mu_k \geq 0$) and $\sum_{j=1}^{k}\mu_j = \sum_{j=1}^{k}\lambda_j$. By Greene's Theorem,

$$\sum_{j=1}^{k-1}\mu_j = \text{maximal size of a union of } k-1 \text{ increasing subsequences in } \bar{\pi} \leq$$

$$\leq \text{maximal size of a union of } k-1 \text{ increasing subsequences in } \pi = \sum_{j=1}^{k-1}\lambda_j.$$

Hence, $\mu_k \geq \lambda_k$. By assumption $(m^k) \subseteq \lambda$, so that $m \leq \lambda_k$. We conclude that there are exactly k parts in μ, and $\mu_1 \geq \cdots \geq \mu_k \geq m$. In other words, $\mu'_1 = k$ and $(k^m) \subseteq \mu'$.

Now, by the second part of Greene's Theorem, $\bar{\pi}$ contains a disjoint union of m decreasing subsequences of total size km. Denote this union by $\hat{\pi}$, and denote its shape by ν. $\hat{\pi}$ is a subsequence of $\bar{\pi}$, hence,

$$\nu'_1 = \text{length of maximal decreasing subsequence in } \hat{\pi} \leq$$

$$\leq \text{length of maximal decreasing subsequence in } \bar{\pi} = \mu'_1 = k.$$

On the other hand,

$$|\nu| = \nu'_1 + \ldots + \nu'_m = km.$$

This shows that the shape of the subsequence $\hat{\pi}$ is $\nu = (m^k)$.

$\square$

5 General Shapes

Theorem 5.1. *For any partition $\mu = (\mu_1, \ldots, \mu_k)$ of m and any positive integer n,*

$$Avoid_n^{\mu} \subseteq \bigcup_{\{\lambda \vdash n \mid (\mu_1^k) \not\subseteq \lambda\}} C^{\lambda}. \tag{5.1}$$

Proof. Let λ be a shape such that $(\mu_1^k) \subseteq \lambda$. By Theorem 4.4, any permutation of shape λ contains a subsequence of shape (μ_1^k). By Theorem 4.1, this subsequence contains a subsequence of shape μ. $\square$

Let $avoid_n^{\mu}$ be the size of the set $Avoid_n^{\mu}$. Theorem 5.1 implies the following asymptotic estimates.

Corollary 5.2. *For any fixed partition* $\mu = (\mu_1, \ldots, \mu_k)$,

$$\limsup_{n\to\infty} (avoid_n^{\mu})^{1/2n} \le ht(\mu) + wd(\mu) \tag{5.2}$$

and

$$\max\{ht(\mu), wd(\mu)\} \le \liminf_{n\to\infty} (avoid_n^{\mu})^{1/2n}, \tag{5.3}$$

where the height of μ $ht(\mu) := \mu_1' - 1$, *and the width of* μ $wd(\mu) := \mu_1 - 1$.

Proof. Let λ be a partition of n, and let f^{λ} be the number of standard Young tableaux of shape λ. By the Robinson-Schensted correspondence

$$(f^{\lambda})^2 = \#\{\pi \in S_n | shape(\pi) = \lambda\}.$$

Combining this fact with Theorem 5.1 we obtain

$$avoid_n^{\mu} \le \#\{\pi \in S_n | (\mu_1^k) \not\subseteq shape(\pi)\} = \sum_{\lambda \vdash n \wedge (\mu_1^k) \not\subseteq \lambda} (f^{\lambda})^2.$$

The asymptotics of the sum on the right hand side was studied by Berele and Regev [BR, Section 7]. By [BR, Theorem 7.21], for fixed μ_1 and k

$$\sum_{\lambda \vdash n \wedge (\mu_1^k) \not\subseteq \lambda} (f^{\lambda})^2 \sim c_1(\mu_1, k) \cdot n^{c_2(\mu_1, k)} \cdot (\mu_1 + k - 2)^{2n}, \tag{5.4}$$

when n tends to infinity. Here $c_1(\mu_1, k)$ and $c_2(\mu_1, k)$ are independent of n. This proves the upper bound (5.2).

For the lower bound, note that by Schensted's Theorem any permutation avoiding (μ_1) also avoids μ. Similarly, any permutation avoiding (1^k) also avoids μ. Thus

$$Avoid_n^{(\mu_1)} \cup Avoid_n^{(1^{\mu_1'})} \subseteq Avoid_n^{\mu}.$$

This implies that (for n large enough; e.g., $n > (\mu_1 - 1)(\mu_1' - 1)$)

$$avoid_n^{(\mu_1)} + avoid_n^{(1^{\mu_1'})} \le avoid_n^{\mu}.$$

Combining this inequality with (5.4) proves the lower bound (5.3).

□

Note: For an evaluation of $avoid_n^{(m)}$ for $m \le 4$ see [St Exer. 7.16(e)]. An asymptotic evaluation of $avoid_n^{(m)}$ for fixed $m > 4$ was first done in [Re].

6 Other Special Cases

6.1 Hooks

In this subsection we show that for hook avoiding permutations and n large enough the "False Conjecture" is correct.

Theorem 6.1. *For any hook $\mu = (m, 1^{k-1})$ and $n > (2m-4)(2k-4)$*

$$Avoid_n^{(m,1^{k-1})} = \bigcup_{\{\lambda \vdash n \mid (m,1^{k-1}) \not\subseteq \lambda\}} C^\lambda.$$

Note: If either $m \leq 3$ or $k \leq 3$ then equality holds for all values of n.

The following analogue of Lemma 4.3 characterizes permutations of hook shape.

Lemma 6.2. *A permutation π has shape $(m, 1^{k-1})$ if and only if π is a union of an increasing subsequence of length m and a decreasing subsequence of length k, intersecting in a unique element.*

Proof. By Schensted's Theorem, a permutation π of shape $(m, 1^{k-1})$ contains an increasing subsequence α with $|\alpha| = m$ and a decreasing subsequence β with $|\beta| = k$, where $|\alpha \cup \beta| \leq |\pi| = m + k - 1$. Since necessarily $|\alpha \cap \beta| \leq 1$, it follows that $|\alpha \cap \beta| = 1$.

The converse follows similarly from Schensted's Theorem.

□

Lemma 6.3. *Let m and k be positive integers.*

(a) If either $m \leq 3$ or $k \leq 3$ then every permutation whose shape contains the hook $(m, 1^{k-1})$ has a subsequence of shape $(m, 1^{k-1})$.

(b) If $m \geq 4$ and $k \geq 4$ then every permutation whose shape contains the hook $(2m-3, 1^{k-1})$ or the hook $(m, 1^{2k-4})$ has a subsequence of shape $(m, 1^{k-1})$.

(c) For any $m \geq 4$ and $k \geq 4$ there exists a permutation whose shape contains $(2m-4, 1^{2k-5})$, but it has no subsequence of shape $(m, 1^{k-1})$.

Note: The results in (a) and (b) above are best possible, as far as the assumed size of a hook contained in the shape is concerned. For (a) this is clear, and for (b) this is the content of (c).

Proof. We shall prove (b); the proof of (a) is similar.

(b) Let π be a permutation whose shape contains the hook $(2m-3, 1^{k-1})$, with $m, k \geq 4$. Then π has an increasing subsequence α of length $2m-3$ and a decreasing subsequence β of length k. If α and β intersect (necessarily in a unique element), then by truncating α to m elements we get by Lemma 6.2 a subsequence of shape $(m, 1^{k-1})$. Otherwise (i.e., assuming that α and β do not intersect) we will show that the union of α and β contains the required subsequence.

Let $\alpha = (\alpha_1, \ldots, \alpha_{2m-3})$ and $\beta = (\beta_1, \ldots, \beta_k)$, so that $\alpha_1 < \ldots < \alpha_{2m-3}$ and $\beta_1 > \ldots > \beta_k$.

Let $ind(\alpha_i)$ denote the index of α_i in the union of α and β (as a subsequence of π); similarly for $ind(\beta_j)$.

Concerning the element α_{m-1} there are three possibilities:

(1) There is an index $1 \leq j \leq k-1$ such that

$$ind(\beta_j) < ind(\alpha_{m-1}) < ind(\beta_{j+1}).$$

(2) $ind(\alpha_{m-1}) < ind(\beta_1)$.

(3) $ind(\alpha_{m-1}) > ind(\beta_k)$.

We shall deal with case (1); the other cases are similar. Since $\beta_j > \beta_{j+1}$, there are now three subcases:

(1a) $\beta_j > \alpha_{m-1} > \beta_{j+1}$.

(1b) $\alpha_{m-1} < \beta_{j+1}$.

(1c) $\alpha_{m-1} > \beta_j$.

In case (1a), α_{m-1} may be added to the decreasing subsequence β, to obtain two intersecting monotone subsequences of lengths $2m-3$ and $k+1$. By truncating these subsequences we will get an increasing subsequence of length m intersecting a decreasing subsequence of length k.

In case (1b), $(\alpha_1, \ldots, \alpha_{m-1}, \beta_{j+1})$ is an increasing subsequence of length m intersecting β.

In case (1c), $(\beta_j, \alpha_{m-1}, \alpha_m, \ldots, \alpha_{2m-3})$ is an increasing subsequence of length m intersecting β.

By Lemma 6.2, in all cases we obtain a subsequence of π having shape $(m, 1^{k-1})$.

(c) The construction extends Example 3.2 (for which $m = k = 4$): take $\pi = (\gamma, \alpha, \delta, \beta)$, where α and δ are increasing sequences of length $m-2$ and β, γ are decreasing sequences of length $k-2$:

$$\alpha = (1, \ldots, m-2); \qquad \beta = (m+k-4, \ldots, m-1);$$

$$\gamma = (m+2k-6, \ldots, m+k-3); \qquad \delta = (m+2k-5, \ldots, 2m+2k-8).$$

It is easy to see that an increasing subsequence of π intersecting γ must be contained (omitting the intersection element itself) in δ, so that its total length is at most $m-1$. Similar analysis of β shows that an increasing subsequence of length m in π must be contained in (α, δ). Analogously, a decreasing subsequence of length k must be contained in (γ, β). The two subsequences cannot intersect.

$\square$

Proof of Theorem 6.1. By Schensted's Theorem, if a permutation π has a subsequence of shape $(m, 1^{k-1})$ then it has an increasing subsequence of length m and a decreasing subsequence of length k. On the other hand, a permutation in $\bigcup_{\{\lambda \vdash n \mid (m,1^{k-1}) \not\subseteq \lambda\}} C^\lambda$ has either no increasing subsequence of length m or no decreasing subsequence of length k. Thus,

$$\bigcup_{\{\lambda \vdash n \mid (m,1^{k-1}) \not\subseteq \lambda\}} C^\lambda \subseteq Avoid_n^{(m,1^{k-1})}.$$

For the other direction, assume that $\pi \in C^\lambda$ with $(m, 1^{k-1}) \subseteq \lambda$. Hence, $\lambda_1 \geq m$ and $\lambda'_1 \geq k$. If either $m \leq 3$ or $k \leq 3$ then, by Lemma 6.3(a), π has a subsequence of shape $(m, 1^{k-1})$. Otherwise (i.e., if $m \geq 4$ and $k \geq 4$), by assumption $(2m-4)(2k-4) < n = |\lambda| \leq \lambda_1 \cdot \lambda'_1$, and therefore either $\lambda_1 > 2m-4$ or $\lambda'_1 > 2k-4$. We can now use Lemma 6.3(b).

$\square$

Corollary 6.4. *For any pair of positive integers m and k, and for $n \geq 4mk$*

$$avoid_n^{(m,1^{k-1})} = avoid_n^{(m)} + avoid_n^{(1^k)} = \sum_{\lambda \vdash n \wedge \lambda_1 < m} (f^\lambda)^2 + \sum_{\lambda \vdash n \wedge \lambda'_1 < k} (f^\lambda)^2,$$

where f^λ is the number of standard Young tableaux of shape λ.

Combining Corollary 6.4 with (5.4) we obtain

Corollary 6.5.

$$\lim_{n\to\infty} (avoid_n^{(m,1^{k-1})})^{1/2n} = \max\{m-1, k-1\}.$$

6.2 Avoiding (2^2)

In this subsection we compute $avoid_n^{(2^2)}$ and show that

$$\lim_{n\to\infty}(avoid_n^{(2^2)})^{1/2n} = \sqrt{2+\sqrt{2}}.$$

In particular, unlike the case of hooks, neither the lower bound nor the upper bound of Corollary 5.2 gives the correct limit in this case.

Example 3.3 shows that for any $n \geq 5$,

$$\bigcup_{\{\lambda\vdash n|(2^2)\not\subseteq\lambda\}} C^\lambda \not\subseteq Avoid_n^{(2^2)}.$$

However, the opposite inclusion does hold.

Proposition 6.6. *For any positive n,*

$$Avoid_n^{(2^2)} \subseteq \bigcup_{\{\lambda\vdash n|(2^2)\not\subseteq\lambda\}} C^\lambda.$$

Proposition 6.6 is a special case of Theorem 4.4. Here we suggest an independent and more informative proof of this result.

Proof. By induction on n. The claim obviously holds for $n \leq 4$. Assume that it holds for $n-1$, for some $n \geq 5$.

For the induction step observe that $C^{(2^2)} = \{2143, 2413, 3142, 3412\}$ consists of all permutations in S_4 for which 1 and 4 are in the 'middle'. It follows that for any permutation π in S_n, if $\pi_1 \notin \{1,n\}$ and $\pi_n \notin \{1,n\}$ then π is not (2^2)-avoiding. Therefore, if $\pi \in S_n$ is (2^2)-avoiding then either

$\pi_1 \in \{1,n\}$ or $\pi_n \in \{1,n\}$. Assume that $\pi_1 \in \{1,n\}$. By the induction hypothesis the shape of the subsequence $(\pi_2,\ldots,\pi_n)$ does not contain (2^2) and is therefore a hook $(r, 1^{n-r-1})$ for some $1 \leq r \leq n-1$. Adding $\pi_1 = 1$ increases the size of the longest increasing subsequence by 1; thus, by Schensted's Theorem the resulting shape is $(r+1, 1^{n-r-1})$. Adding $\pi_1 = n$ increases the size of the longest decreasing subsequence by 1; again, by Schensted's Theorem the resulting shape is $(r, 1^{n-r})$. The case $\pi_n \in \{1,n\}$ is similar.

$\square$

Corollary 6.7. *For any positive integer n*

$$avoid_n^{(2^2)} = \frac{1}{2}(2+\sqrt{2})^{n-1} + \frac{1}{2}(2-\sqrt{2})^{n-1}.$$

Proof. It follows from the proof of Proposition 6.6 that

$$avoid_n^{(2^2)} = 4 \cdot avoid_{n-1}^{(2^2)} - 2 \cdot avoid_{n-2}^{(2^2)}.$$

The solution of this linear recursion (with appropriate initial values) gives the desired result.

□

7 Final Remarks and Open Problems

7.1 Algebraic Structure

Let R be the set of all representatives of minimal length of left cosets of S_m in S_n (length here, as usual, is in terms of the Coxeter generators, i.e., adjacent transpositions). For any partition μ of m, the set C^μ of all permutations of shape μ is a two-sided Kazhdan-Lusztig cell in S_m. For any $n \geq m$ the set of all permutations in S_n which are not μ-avoiding coincides with the set $RC^\mu R^{-1}$. Theorem 5.1 claims that for hook shapes the set $RC^\mu R^{-1}$ is a union of two-sided Kazhdan-Lusztig cells. This phenomenon generalizes a beautiful well-known fact: The set RC^μ (or: $C^\mu R^{-1}$) is a union of Kazhdan-Lusztig left (resp. right) cells [Sr, BV Prop. 3.15]. See also [GaR, Ro]. Barbasch and Vogan gave an algebraic proof of this fact by associating the set RC^μ to induced representations. An algebraic interpretation for the results in this paper is required. These and other relations with representation theory deserve further study.

7.2 Asymptotics

Regev calculated, by considering Schensted's Theorem, the exact asymptotics of $avoid_n^{(m)}$ [Re]. In this paper we have generalized this "RSK approach" to prove that for any partition μ there exists a constant $c(\mu)$ such that, for any n,

$$avoid_n^{\mu} \leq c(\mu)^n.$$

Note that from Corollary 5.2 and Corollary 6.7 it also follows that, for μ not strictly contained in (2^2), there exists a constant $\tilde{c}(\mu) > 1$ such that $avoid_n^{\mu} \geq \tilde{c}(\mu)^n$ for n large enough.

A far reaching generalization was conjectured by Stanley and Wilf [Bo1].

The Stanley-Wilf Conjecture. *For any fixed permutation σ there exists a constant $c(\sigma)$ such that, for any n*

$$\textit{avoid}_{\,n}(\sigma) \leq c(\sigma)^n,$$

where $\textit{avoid}_{\,n}(\sigma)$ is the number of all σ-avoiding permutations in S_n.

By a result of Arratia [Ar], if this conjecture holds then actually the limit $\lim_{n\to\infty} \textit{avoid}_{\,n}(\sigma)^{1/n}$ always exists (and is finite).

The Stanley-Wilf conjecture holds for all $\sigma \in S_3$ [K, p. 238] and all $\sigma \in S_4$ [Bo1, Bo2], as well as for many other cases (see [SSi], [Bo3] and their references). Recently, Alon and Friedgut [AF] have applied Davenport-Schinzel sequences to prove a somewhat weaker version of the conjecture for arbitrary σ. An interesting challenge is to apply the "RSK approach" to attack the Stanley-Wilf Conjecture; namely, to apply Greene's Theorem and methods presented in this paper to sets avoiding a single permutation.

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